Conservation statements, scaling approaches, and the adjustment of rotating fluids

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This worksheet draws heavily from Pedlosky (2003) Chapter 12 and Vallis (2006) §3.8.

Imagine a fluid layer of average height H, constant density ρ, rotating with *constant* angular velocity about the vertical axis z. We will examine a one-dimensional "dam-break" problem. The initial height of this constant-density ocean is $\eta = -\eta_0 \,\mathrm{sgn}(x)$, shown in Figure 1. There, the x-axis is the horizontal dimension x normalized by a length scale L_D termed the deformation radius (we will see why this is done later). With the exception of L_D, subscripts indicate differentiation.

What happens next?

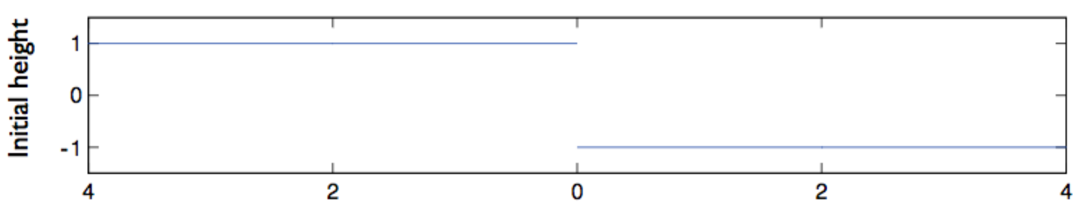

Figure 1: Initial sea surface height. x-axis is x/L_D.

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1. Start with the following equations.

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial P}{\partial x} \tag{1}$$
$$\frac{\partial v}{\partial t} + fu = -\frac{\partial P}{\partial y} \tag{2}$$
$$0 = -\frac{\partial P}{\partial z} - g \tag{3}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

   $P = p/\rho$. List the approximations we have made.

2. Solve the equation for pressure P. What is the appropriate boundary condition?

3. Substitute into the two horizontal momentum equations. Which of the dimensions (x, y, z, t) does the right hand side i.e. the *forcing* terms depend on? What does this tell you about u, v?

4. Integrate the continuity equation in z. Again, apply appropriate boundary conditions.

5. We have reduced a system of 4 PDEs to 3, but that is still too much. Let's go further. Eliminate *both* pressure terms from your two horizontal momentum equations. This results in an equation for vorticity $\zeta = v_x - u_y$.

6. Now down to two equations but we can go even further! Combine your vorticity equation and integrated continuity equation you derived in step 4. Does the resulting equation look like a conservation statement?

7. The conserved quantity is *linearized shallow-water* potential vorticity PV generally denoted by q. PV takes many forms depending on the system of equations you start off it. Does *this* PV equation have wave solutions?

8. Let's form a wave equation. Earlier, you took the *curl* of the horizontal momentum equations to form vorticity ζ. Instead, let's try the *divergence* to form an equation for $u_x + v_y$. After some algebra,

$$-\frac{1}{gH}\eta_{tt} + \nabla_H^2\eta - \frac{f^2}{gH}\eta = \frac{1}{g}\nabla_H^2 P_a(x,y,t) - \frac{f}{g}q(x,y) \tag{5}$$

Note: Using the curl & divergence of the momentum equations and then using the continuity equation is a standard strategy to reduce even the non-linear Navier Stokes equations.

What are the consequences of q being a function of only (x, y)? Can you decompose η into two parts? How do they differ?

9. Scale and obtain an approximate wave speed from this wave equation?

10. For atmospheric pressure $P_a = 0$, write the equations governing the *steady-state* solution. Do this for (5) and the two horizontal momentum equations viz. (1) and (2).

    Are (u, v) in geostrophic balance at $t = \infty$?

11. Simplify your wave equation for η given the one dimensional initial condition in Figure 1.

12. Why solutions do you expect from this simplified wave equation? Sinusoidal? Exponential? Something more complicated?

    What are the appropriate boundary conditions for $x \longrightarrow \pm\infty$? Without solving the differential equation, can you guess the "length scale" of the final solution? Try balancing two terms. What is the implication of the existence of this length scale?

13. The solution for equilibrium SSH η is

$$\eta = \begin{cases} -\eta_0(1 - e^{-x/L_D}),\ x > 0 \\ \ \ \eta_0(1 - e^{x/L_D}),\ x < 0 \end{cases} \tag{6}$$

$$L_D = \frac{\sqrt{gH}}{f_0} = \frac{\text{(characteristic wave phase speed)}}{f} \tag{7}$$

is the Rossby radius of *deformation*. (I have renamed L_x to L_D)

There is a well-defined region of adjustment. L_D is a length scale describing that adjustment region.

From Pedlosky's Waves book:

> The deformation radius is an intrinsic length scale and measures the tendency for gravity to smooth disturbances out horizontally against the tendency for rotation to link the fluid together vertically along the rotation axis.

How would the *non-rotating* problem differ? Like when a levee breaks?

14. See Figure 2 for the steady-state solution to the dam break problem (Vallis, 2006, Fig. 3.10). Are η and v non-zero for $x > L_D$. Why or why not?

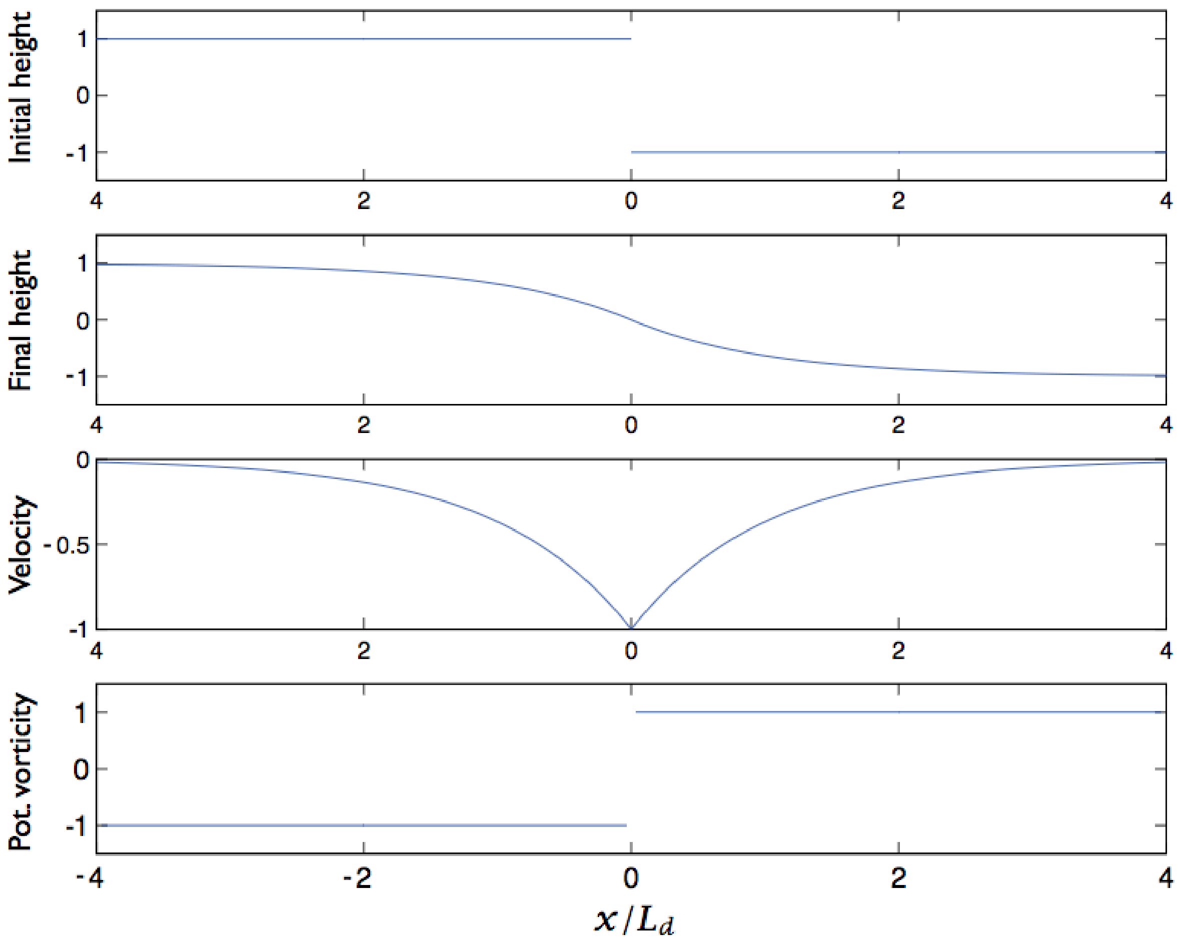

Fig. 3.10 Solutions of a linear geostrophic adjustment problem. Top panel: the initial height field, given by (3.121) with $\eta_0 = 1$. Second panel: equilibrium (final) height field, η given by (3.134) and $\eta = f_0\psi/g$. Third panel: equilibrium geostrophic velocity (normal to the gradient of height field), given by (3.135). Bottom panel: potential vorticity, given by (3.130), and this does note evolve. The distance, x is non-dimensionalized by the deformation radius $L_d = \sqrt{gH}/f_0$, and the velocity by $\eta_0(g/f_0L_d)$. Changes to the initial state occur only within $\mathcal{O}(L_d)$ of the initial discontinuity; and as $x \to \pm\infty$ the initial state is unaltered.

Figure 2: Figure 3.10 from Vallis (2006)

15. q is conserved and hence *does not change*. This constraint determines the final form of η and the balanced geostrophic velocity field v. From Vallis (2006),

> in the linear approximation, geostrophy is the minimum-energy state for a given field of potential vorticity.

16. The system of equations can be manipulated to form an energy equation:

$$\frac{\partial}{\partial t}\left[H\left(\frac{u^2+v^2}{2}\right)+\frac{g\eta^2}{2}\right]+\nabla\cdot(g\eta\hat{u})H=0 \tag{8}$$

What are the initial and final potential & kinetic energies of the system? What happens to the difference in total energies? Use the solution for η above and the following solution for (u,v)

$$u=0 \qquad v=-\frac{g\eta_0}{fL_D}e^{-|x|/L_D}$$

17. Pedlosky examines the original Rossby (1938) adjustment problem. The initial condition is a jet of width a and uniform sea surface height. At $t = 0$

$$v = \eta = 0$$

and

$$u = \begin{cases} U, & |y| < a \\ 0, & |y| > a \end{cases} \tag{9}$$

The underlying equations are the same viz. those in step 1. Discuss various aspects of the adjustment *without* solving the whole problem in detail.

a) How will the initial distribution of u change?

b) What happens to v?

c) What happens to η?

d) Will you have two adjustment regions? Or one like in the previous problem? What is the length scale of the adjustment region?

18. For this new adjustment problem, one can show that the ratio of total energy (TE) of the final state TE_F to initial total energy TE_I is

$$\frac{TE_F}{TE_I} = \frac{1 - e^{-2a/L_D}}{a/L_D}.$$

Draw what this function looks like and discuss the limits $a \ll L_D$ and $a \gg L_D$.

19. Key points:

    a) The conservation of PV is extremely important and useful tool in GFD.

    b) In a *rotating* fluid, variations in height and field are *not* radiated to infinity. There is an adjustment region that reflects the initial condition.

    c) The Rossby radius of deformation is an important length *scale* that defines the adjustment region. The appropriate Rossby radius for a system generally determines the spatial extent[1] over which a fluid directly responds to changes in forcing.

    d) Waves unleashed by the adjusting fluid are not restricted by the deformation radius and radiate away to infinity (unless the fluid is in a box).

20. Looking ahead:

    a) The code I used for the movies is available here with documentation: `https://empslocal.ex.ac.uk/people/staff/gv219/codes/geoadjust.html` . I encourage you to experiment.

       If you have a python distribution setup, running `python linearshallowwater.py` on the command-line should get it running.

    b) Videos from Jamie Pringle (UNH): `http://oxbow.sr.unh.edu/WaveMovies/` These videos are great! They show what happens for similar adjustment problems but in a slightly more realistic ocean where the rotation frequency f is not a constant but instead varies linearly with latitude y as $f = f_0 + \beta y$. In that case, internal waves are *not* the only possible waves; there are others, some of which carry PV, and so the ocean adjusts quite differently.

    c) Vallis (2006) §3.8 is interesting reading.

    d) Gill - Atmosphere-Ocean Dynamics: Gill treats the simple adjustment problem we just did, but then extends it over many chapters to include stratification, β-plane oceans. The results are analogous to what we learned here; but very interesting.

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[1] upto an O(1) factor