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ANALOGIES OF OCEAN/ ATMOSPHERE ROTATING FLUID DYNAMICS WITH GYROSCOPES TEACHING OPPORTUNITIES

BY THOMAS W. N. HAINE AND DEEPAK A. CHERIAN

Aspects of geostrophic flow and inertial oscillation in the ocean and atmosphere are analogous to the motion of a rapidly spinning gyroscope.

The dynamics of the rotating shallow-water (RSW) system are fascinating and counterintuitive. Small-amplitude motion consists of a mixture of two dynamical modes. One is called geostrophic flow, which has the lowest frequency. The other is called the inertiagravity wave, which can take any frequency above a certain threshold. Inertia-gravity waves at the threshold frequency are called inertial oscillations and are controlled by rotation. Geostrophic flow and inertial oscillations are ubiquitous in more complex fluids, such as Earth's ocean and atmosphere. Thinking about these dynamical modes can be a challenge because the influence of rotation is unfamiliar to many people. Building intuition about geostrophic flow and inertia-gravity waves in the RSW system is important, however, because it provides a solid foundation for understanding more complex motions in real environmental fluids.

This article concerns a simple tangible device that has analogous dynamics to the RSW system: the gyroscope, or spinning top. Rotating rapidly, the gyroscope seems to defy gravity by not falling over, as long as it possesses sufficient spin. The dynamical analogy between a rotating fluid and a gyroscope is an interesting and novel finding that we explain here.¹ The gyroscope may be valuable in the classroom ▶

"Gyroscopic pumping" is sometimes discussed in the atmospheric dynamics and astrophysics literature, but that refers to meridional overturning driven by azimuthal stresses (e.g., Haynes et al. 1991). The National Aeronautics and Space Administration (NASA) *Terra* satellite image 1250 UTC 21 June 2004, courtesy of Jacques Descloitres, MODIS Rapid Response Team, NASA/GSFC. Cloud (at various altitudes), land ice, and sea ice are visible in white. The swirling green streaks in the ocean are blooms of high phytoplankton concentration, most likely coccolithophores. The swirls reveal the geostrophic turbulence in the surface ocean currents. TABLE I. Similarities and differences between ocean/atmosphere rotating fluid dynamics and the gyroscope. The specific mathematical link is given by Eq. (7) and concerns the linear, longwave limit of rotating shallow-water flow (see the supplement for mathematical details). Ocean/atmosphere fluid dynamics are more complicated than gyroscope dynamics. Cross references to the main text and the online resources are indicated.

Similarities ("rotating fluids are analogous to gyroscopes")	Ocean/atmosphere rotating fluid dynamics ("geostrophic flow and inertial oscillation in rotating fluids")	Gyroscope ("gyroscope nutation and precession")
Phenomenology	Supports permanent pressure gradients and currents	Does not topple over when spinning
Low-frequency forced motion	Geostrophic flow (Figs. 1, 3, geostrophic_movie.mov)	Precession (Figs. 6, 7, gyroscope_schematic_ movie.m4v, gyroscope_movie.m4v)
High-frequency free motion	Inertial oscillation (Figs. 2, 4, inertial_osc_movie.m4v)	Nutation (Figs. 6, 7, gyroscope_schematic_ movie.m4v, gyroscope_movie.m4v)
Response to impulse	Transient inertial oscillations with frequency proportional to rotation rate and amplitude (inertial circle radius) inversely proportional to rotation rate	Transient nutation with frequency proportional to spin and amplitude inversely proportional to spin
Response to steady force	Geostrophic flow at 90° to force with speed proportional to force and inversely proportional to rotation rate	Precession at 90° to force with speed proportional to force and inversely proportional to spin
Nondimensional number measuring rotational effects	Rossby number, Eq. (1)	Gyroscope number, Eq. (4)
Small-amplitude dynamics	Eq. (5)	Eq. (6)
Minimum energy state	Pure geostrophic flow	Steady precession
Differences ("discussion of the analogy and differences")		
Form of Coriolis term	Clockwise inertial circles in Northern Hemisphere, beta effect	Counterclockwise nutation, no analog to beta effect
Spatial variations	Has spatial variations in flow and fluid properties	No analog to spatial variations

as it may be used in demonstrations to illustrate RSW concepts that underpin large-scale motions in the ocean and atmosphere. The gyroscope is less useful as a tool to explain the basic physics of RSW

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In final form 23 September 2012 ©2013 American Meteorological Society flow. The physics of gyroscopic motion is analogous to particular aspects of RSW flow but has a distinct character of its own. Instructors seeking to explain the reasons for geostrophic flow and inertial oscillations in rotating fluids should focus on explaining the Coriolis effect. The Coriolis effect can be derived mathematically by a simple coordinate transformation into Earth's rotating frame of reference, and that is typically presented in the classroom.

This approach is unsatisfying to many people, however (see, e.g., Stommel and Moore 1989). Simple physical arguments to explain the Coriolis effect consider motion in the nonrotating frame. In this vein, Persson (1998) provides an historical account and many useful references. Phillips's (2000) explanation proceeds from dynamical principles familiar from elementary mechanics and also addresses geostrophic flow and inertial motion. Durran (1993) analyzes inertial motion in detail and clarifies the origin of the RSW equations by considering a physical model of a frictionless particle moving in a parabolic dish (see also Durran and Domonkos 1996). Among textbook accounts, Marshall and Plumb (2008) is comprehensive and accessible because it builds intuition via demonstrations with rotating tables. The present analogy between RSW dynamics and gyroscopes is not intended to replace these explanations of the basic physics behind the Coriolis effect.

We begin by reviewing geostrophic flow and inertial oscillations in the ocean and atmosphere. Examples of these modes are also presented in a rotating



Fig. 1. Geostrophic motion. (a) Atmospheric cyclone over the Irminger Sea and southwest Iceland at 1410 UTC 4 Sep 2003, as seen by the NASA Aqua satellite and visualized by clouds. The diameter of the system is around 750 km. (b) Surface ocean geostrophic turbulence as seen by streaks of green phytoplankton south of Iceland at 1250 UTC 21 Jun 2004 by the NASA Terra satellite. The diameter of the cyclone in the center of the frame is about 40 km.

tank of water. Next the gyroscope is introduced and explained. Examples of gyroscopic motion are shown. We explain the analogies between the gyroscope and RSW dynamics in the following section. Finally, the limits to the analogy are discussed: it applies to specific aspects of geostrophic and inertial motion because important fundamental differences exist between each system. Table 1 summarizes this comparison and provides linkage to the dynamics of the oceans and atmosphere relevant to higher education. Four movies that demonstrate these principles and analogies are provided in the online auxiliary material (http://dx.doi.org/10.1175/ BAMS-D-12-00023.3). Mathematical details in the main text are kept to a minimum, but they appear in an online supplement (http://dx.doi.org/10.1175/BAMS -D-12-00023.2) for clarity, as a resource template for educators, and for further study.

GEOSTROPHIC FLOW AND INERTIAL OSCILLATION IN ROTATING FLUIDS. The

extratropical ocean and atmosphere almost always exhibit geostrophic flow and inertial oscillation at large scales. For example, Fig. 1 shows satellite images of geostrophic winds and currents, both near Iceland. In Fig. 1a, the air swirls cyclonically (counterclockwise) around a low pressure center in the middle of the frame. The force due to the pressure gradient is directed toward this low pressure center, but the wind circles around it, nearly at right angles, and must complete several loops before reaching the center. In Fig. 1b, the surface ocean currents are seen through their action on green phytoplankton streaks. A counterclockwise vortex (an oceanic cyclone) is visible in the middle of the image as a spiral streak, similar to the atmospheric cyclone. Again, there is a low pressure center at the core of this vortex. The scales of motion are different in the two cases: the cyclone diameter is about 750 km in the atmosphere and about 40 km in the ocean. The time scales are different as well: a day or two for the atmosphere² and a week or two for the ocean. Nevertheless, the fundamental dynamics in each case are the same. The flow is mainly geostrophic and dominated by Coriolis forces due to Earth's rotation.

Geostrophic flow occurs when the nondimensional Rossby number is small (and dissipative effects are negligible). The Rossby number is

$$Ro = \frac{V}{fL} \tag{1}$$

for flow speed *V*, length scale *L*, and Coriolis parameter *f*, equal to twice the rate of rotation about the vertical axis. In the periphery of the cyclones show in Fig. 1, *Ro* is small and the flow is geostrophic. Toward the centers of the cyclones the Rossby number is larger. There, some of the inward pointing pressure-gradient force provides the centripetal

² The atmospheric cyclone in Fig. 1a is a polar low, a small, short-lived depression [for another example and further details, see Fig. 1.16 in Rasmussen and Turner (2003)]. Many atmospheric cyclones have larger scales and a life cycle of several days to a few weeks, however.



FIG. 2. Near-inertial motion of satellite-tracked surface drifters in the North Pacific Ocean (from van Meurs 1998).

force required for the fluid to flow around the low pressure center. When centripetal and Coriolis forces are similar in magnitude, the Rossby number is near one (called gradient wind balance). Even closer to the cyclone center, the centripetal force dominates the Coriolis force and the Rossby number is large (called cyclostrophic balance).

Figure 2 is a good example of (near-) inertial oscillation in the North Pacific Ocean. A storm excites inertial oscillations over an area 350 km



on a side (van Meurs 1998). The surface water moves in clockwise circles for 3 weeks, as shown by the tracks of satellite-tracked surface drifters. The drifter tracks are coherent and in phase over most of the area shown (the phase varies only by about a quarter period in total). That is, the drifters all move north together, then all turn east, then all turn south, and so on. The period of the oscillation is about 19 h. The slow drift of the trajectories is due to geostrophic eddies like those in Fig. 1b. This low-frequency geostrophic motion is much more variable in space (nearby cycloids move in different directions). The in-phase circular movement is an inertial oscillation, however, which is distinctly different. In particular, the inertial oscillation is more predictable than the geostrophic eddies

because it involves mainly linear dynamics. The motion of the geostrophic eddies involves nonlinear dynamics that can cause chaotic effects that make long-term predictions very hard. The coexistence of inertial oscillations and geostrophic eddies is an interesting property of ocean and atmosphere fluid dynamics.

Inertial oscillations are also present in the atmosphere. For example, the nocturnal jet is a low-level diurnal flow that has been explained as an inertial oscillation (Blackadar 1957). During the day the atmospheric boundary layer is actively mixed by convection owing to solar heating at the ground. At dusk, the convection intensity falls and the related frictional forces disappear. The balance of forces changes and air parcels experience an unbalanced Coriolis force that causes the flow to veer anticyclonically in an inertial oscillation. Evidence of inertial oscillations of this kind is not widespread, however, and observed in-

FIG. 3. Geostrophic motion in a rotating tank of water used for classroom demonstrations of atmosphere and ocean fluid dynamics (the Weather in a Tank equipment: see Illari et al. 2009). The image, taken from above, shows streaks of dye in a tank of water 10 cm deep, rotating at 20 rpm. The diagonal measures about 17 cm. The black dot is a floating piece of paper from a hole punch. Only part of the 40-cm square tank is shown. See geostrophic_movie.mov in the online auxiliary material. ertial oscillations appear to be more commonly associated with fronts rather than sunset (Lundquist 2003).

Geostrophic flow and inertial oscillations are also easily shown using a rotating laboratory turntable. Figure 3 shows such an example of geostrophic flow using a portable apparatus that is designed for classroom demonstrations (the "Weather in a Tank" apparatus; Illari et al. 2009).³ A dipole eddy is visible. Flow in the lower (upper) vortex is counterclockwise (clockwise). The current is associated with a depression in the water surface in the lower vortex (low pressure) and an elevation in the upper one (high pressure). In subsequent frames the black dot, a floating piece of paper, moves between the two eddies.

Geostrophic flow is the dynamical balance between a pressuregradient force and a Coriolis force. In Fig. 3, the pressure-gradient force acts from the high pressure center to the low pressure center. The Coriolis force is a fictitious deflective force that arises because of the rotation of the reference frame. It is proportional to flow speed and directed at right angles to both the rotation axis and the current direction. In Fig. 3,

it opposes the pressure-gradient force as shown. The Coriolis force is associated with the geostrophic current at right angles visualized by the black dot and the colored dye.

In Fig. 3, the pressure gradient is due to changes in the water level η in the tank (similarly, the pressure gradient in Fig. 1b is also due to changes in sea level). The geostrophic current speed equals -M/f, where $M = -g\partial\eta/\partial s$ for distance *s* measured from the lower eddy to the upper one (*g* is gravitational acceleration). The change in water level between the eddies is only about 40 μ m, visible with special optical altimetry equipment (Rhines et al. 2007) but not with the naked eye (in Fig. 1b the sea level differences are 10–20 cm). The counterintuitive hallmark of geostrophic fluids is to support these pressure gradients over long periods of time.



Fig. 4. Inertial oscillation in a rotating tank of water at 20 rpm. (top) Snapshot of four floating particles (A–D) with particle locations tracked over time (small black dots). An impulsive wind stress is applied as indicated by blowing on the surface between approximately 8 and 9.5 s. Some gravity waves are also excited with crests traveling in the direction of the stress but they rapidly disappear. The particle locations are found every frame from a video of the experiment taken at 30 frames per second. The dimensions of the image are approximately 7 cm by 23 cm and the water depth is 10 cm. (bottom) Time series of the motion of the four particles. See inertial_osc_movie. m4v in the online auxiliary material.

Figure 4 shows an example of inertial oscillation using the laboratory turntable.⁴ Initially there is no flow. The experimenter blows over the water surface for a second or two, as indicated. The particles then execute inertial oscillations with a period of 1.5 s for the tank rotation rate of 20 rpm. Notice how the motion is spatially uniform and in phase, so that peaks in the y^* position in the lower panel are synchronized, similar to Fig. 2. There is no horizontal variation in the current under inertial oscillation, unlike geostrophic flow.

Inertial oscillation is a temporally oscillating flow of a rotating fluid in the absence of pressure-gradient forces. Particles in this motion trace horizontal circular paths. Inertial oscillation occurs from the continuous deflection of moving fluid by the associated Coriolis force, which is perpendicular to the

³ See also the online movie geostrophic_movie.mov.

⁴ See also the online movie inertial_osc_movie.m4v.

velocity. The period of inertial oscillations equals $2\pi/f$, or T/2 here for turntable rotation period T, consistent with the observed period of 1.5 s. Notice that the period of inertial oscillations decreases as the rotation period decreases. Likewise, the frequency of inertial oscillation is equal to the Coriolis parameter, which is proportional to the rotation rate.



Fig. 5. Photos of gyroscopes. (top left) The Massachusetts Institute of Technology and the AC Spark Plug Division of General Motors (MITAC) mechanically driven, gimbal-mounted gyroscope, shown also in Fig. 7. (top right) Brass gimbal-mounted lecture gyroscope. (bottom) Selection of freestanding toy gyroscopes.

GYROSCOPE NUTATION AND PRECES-SION. Geostrophic flow and inertial oscillation, the two fundamental modes of RSW dynamics, are analogous to the motion of a gyroscope. Before describing this connection in detail, the fundamental modes of a gyroscope are explained.

The spinning gyroscope behaves differently to the resting gyroscope. A freestanding gyroscope with no spin simply topples over in a familiar way. A rapidly spinning gyroscope does not. There are two unexpected behaviors, which are called precession and nutation (as shown in Fig. 6).⁵ Precession is the progressive movement of the gyroscope spindle under torque: the spindle traces out a cone (left panel of Fig. 6). Two common configurations provide the torque. The simplest option rests the gyroscope spindle on a flat surface (Fig. 5, bottom; Fig. 6). The gravitational force acting through the center of mass of the freestanding gyroscope and the normal reaction at the surface then furnish the torque. For a gimballed gyroscope, torque is applied by pushing on the spindle: for example, with a finger or a weight (Fig. 5, top; Fig. 7). Despite these differences in configuration, the gyroscope's response to a gravitational torque is essentially the same: the free tip of the gyroscope traces out a horizontal circle (Fig. 6). This means that the spindle traces out a vertical cone with apex at the fixed point. With no spin, the gyroscope topples over with the free tip moving in a vertical plane. This is the most unexpected property of a spinning gyroscope: it responds to a push by moving at right angles.6



Fig. 6. Schematic diagrams of a freestanding gyroscope: (left) precession (motion under torque), (middle) nutation (torque-free precession), and (right) precession and nutation together. See gyroscope_schematic_ movie.m4v in the online auxiliary material.

⁵ See also the online movie gyroscope_schematic_movie.m4v.

⁶ Multiple physical explanations for precession exist. They include conservation of vector angular momentum (French 1971; Butikov 2006) and consideration of the forces on point masses comprising the gyroscope rotor (Edwards 1977; Eastman 1975; Barker 1960).

Nutation-from "nodding" in Latin—is an oscillatory wobble of the gyroscope's spindle (middle panel of Fig. 6). Nutation is not caused by a torque: hence the alternative name torque-free precession. It arises when the total angular momentum of the gyroscope does not equal the angular momentum about the gyroscope's spindle. In other words, the gyroscope is simultaneously spinning about the spindle axis and also another axis. Both freestanding and gimballed gyroscopes nutate. Nutation is seen most easily in the gimballed gyroscope because then no gravitational torque applies. There is no precession and knocking the spindle causes pure nutation. The nutation shown in Fig. 6, as well as in the online movie, shows a pure nutation.

An American football is a familiar object that is often seen nutating. When thrown correctly, the ball only spins about its axis of symmetry. Then no wobbling of the ball occurs (no nutation). Thrown incorrectly, the axis of symmetry traces out a cone with respect to the center of mass and the ball wobbles rapidly in the air. In this case, the total angular momentum vector contains a com-

ponent that is not along the symmetry axis.⁷

Usually, gyroscopes exhibit both precession and nutation together. In the case of a freestanding gyroscope, the spindle tip loops in a cycloid (right panel of Fig. 6). The case of a gimballed gyroscope under a gravitational torque from a weight is shown in Fig. 7.⁸ Both nutation and precession are observed. Nutation is the high-frequency circular motion of the gyroscope tip, and precession is the low-frequency sweep of the tip in the horizontal plane. Taken together, the tip motion is cycloidal. Friction in the gimbal bearings causes the nutation to slowly die away as seen in the bottom panel of Fig. 7. The precession lasts much longer, although friction eventually causes the gyroscope tip to point down as it would in the absence of spin. The force of the weight in Fig. 7 is directed



FIG. 7. Motion of a gyroscope under torque. The mechanically driven gyroscope is a gimbal-mounted MITAC model rotating at 225 rpm. (top) A weight (visible behind the conical tip) is attached to one end to generate a gravitational force, hence a torque, and the motion is started by a firm push of the hand with the tip pointing left. The white line traces the motion of the tip of the gyroscope over 12 s (to track the tip, the gyroscope is lit by ultraviolet light that reveals only the tip itself; the background image is from an otherwise identical case, immediately before). Nutation is the circular counterclockwise motion of the tip; precession is the steady horizontal motion to the right in response to the force. The cycloidal tip motion is due to a sum of precession and nutation. The slow decay in nutation amplitude is caused by friction in the gimbal bearings. (bottom) Vertical position of the gyroscope tip over time. See gyroscope_movie.m4v in the online auxiliary material.

to move the tip downward (the torque is horizontal), but the gyroscope responds by precessing at right angles in the horizontal direction (the precession vector is vertical).

Before discussing the link to geostrophic flow and inertial oscillation, a few details are worth noting. First, both precession and nutation are oscillations that cause the gyroscope's spindle to trace out cones (Fig. 6), but their frequencies differ. For precession, increasing the flywheel spin causes the frequency (spindle-tip speed) to decrease, whereas for nutation it increases the frequency. The formulas are

$$\omega_{p} = \frac{mgl}{I_{3}\omega_{3}}$$
 (precession frequency) (2)

and

⁷ For physical explanations of nutation, see Butikov (2006), French (1971), and Lock (1989). For a discussion of American football dynamics, see Brancazio (1987).

⁸ See also the online movie gyroscope_movie.m4v.

$$\omega_{n} = \frac{I_{3}\omega_{3}}{I}$$
 (nutation frequency), (3)

where mg is the weight driving the precession, applied at a horizontal distance l from the gyroscope's fixed point (so mgl is the torque); I_3 is the moment of inertia about the spindle, I is the moment of inertia perpendicular to the spindle, and ω_3 is the angular velocity about the spindle.

Second, Eq. (2) shows that the precession frequency (angular velocity) is proportional to the applied force *mg*. This fact is another surprising property of the precessing gyroscope. Simple Newtonian dynamics says that an applied force is proportional to acceleration of a body in the direction of the force. The toppling gyroscope with no spin behaves like this. The spinning gyroscope seems totally different because the force is proportional to the speed of the body, an Aristotelian idea. The naive intuition based on Newtonian dynamics (ignoring the flywheel spin) is therefore qualitatively wrong!

Third, a related question is why a spinning gyroscope does not topple over. An energy analysis

GYROSCOPES AND TOPS

The gyroscope, or top, is a simple device that takes many forms. The key common elements are a heavy flywheel, which can turn rapidly on a spindle (axis of symmetry; Fig. 5). There is a mechanism to give the flywheel a large angular velocity. A toy top is spun by finger and thumb, for instance, or a thread is wrapped round the spindle and pulled away. Tops spin while the spindle is supported by a solid object like a table. Gyroscopes also have bearings and a cage to support the flywheel and spindle. The bottom panel of Fig. 5 shows examples of two freestanding gyroscopes and a top.

More sophisticated gyroscopes have a mechanism called a gimbal that allows the spindle axis to move freely. The spindle can point in any direction, while the center of mass remains fixed. Industrial gyroscopes, used in inertial navigation instruments in aircraft and ships, for instance, operate this way. They also have a motor to keep the gyroscope spinning. The Massachusetts Institute of Technology and the AC Spark Plug Division of General Motors (MITAC) gyroscope in the top-left panel of Fig. 5 is an example. The brass lecture gyroscope in the topright panel of Fig. 5 uses a rubber wheel on an electric twist drill to spin it. The flywheel can reach speeds greater than 7000 rpm and will spin for more than 20 min.

Gyroscopes and tops are engaging devices that have fascinated curious children for generations. Spinning gyroscopes behave in a counterintuitive way that is quite different from their behavior with no spin. illuminates this issue (see the supplement for details). The gravitational potential energy is a relatively small contribution to the total gyroscope energy when it spins fast. A much larger effective potential energy exists that depends on the spin. This potential energy is quadratic in the topple angle and forms a potential well (Fig. S2 in the supplement). As the gyroscope topples over it gains kinetic energy. This kinetic energy is traded with the effective potential energy. From this point of view, nutation can be viewed as oscillations inside the potential energy well. The nutating gyroscope is trapped inside the potential well and cannot topple. The well becomes narrower as the gyroscope spins faster so that the nutation amplitude decreases with increasing spin.

Finally, note that all of these phenomena apply to rapidly rotating gyroscopes. More precisely, they apply to gyroscopes that possess large angular momentum about their spindles, compared to the angular momentum if the gyroscope was swinging like a simple pendulum without spin. (To imagine the simple pendulum, think of the nonrotating gimballed gyroscope in Fig. 7 with the weight making it swing.) A nondimensional number measures the sizes of these two angular momenta. It is

$$\varepsilon = \frac{I\omega_{_{pe}}}{I_{_{3}}\omega_{_{3}}},\tag{4}$$

where the simple pendulum frequency is $\omega_{pe} = \sqrt{mgl/I}$. This number is 0.0538 for the gyroscope in Fig. 7, indicating the dominance of spin angular momentum (see also Fig. S3 in the supplement).

ROTATING FLUIDS ARE ANALOGOUS TO GYROSCOPES. Having explained the fundamental modes of gyroscope motion, the link to rotating fluids can be stated (see also Table 1).

Geostrophy is analogous to precession. The analogy can be seen several ways. In both cases, the system responds to a steady force by moving at right angles and not in the direction of the force. The speed of motion, not the acceleration, is proportional to the force. This means that the motion stops when the force disappears. The speed is inversely proportional to the rotation rate [think of the geostrophic formula and the precession Eq. (2)]. The fluid analog of the nondimensional gyroscope number ε is the Rossby number, which is explained in the previous section on geostrophic flow and inertial oscillation in rotating fluids. Small ε and small Rossby number mean that rotation dominates the dynamics (the Rossby number is about 10⁻³ for the rotating tank in Figs. 3 and 4).

Inertial oscillation is analogous to nutation. In both cases the frequency is proportional to the rotation rate [think of the inertial oscillation formula and the nutation Eq. (3)]. Similarly, when the amplitude of oscillation is measured by the radius of the trajectory, the amplitude is inversely proportional to the rotation rate. In other words, faster spin means faster smaller inertial circles and nutations. Moreover, the spectral gap between the high- and low-frequency modes grows with rotation rate. For the rotating fluid the ratio of inertial frequency to the inverse geostrophic advection time scale grows as Ro⁻¹ (where V/L defines the inverse advection time scale; if L denotes the length of a periodic domain, V/L is the frequency at which a floating particle returns to the same spot). For the gyroscope, the ratio of nutation frequency to precession frequency grows as ε^{-2} .

More formally, the analogy connects gyroscope dynamics and RSW flow in the longwave limit (vanishing wavenumber). Small-amplitude (i.e., small Rossby number) RSW dynamics are commonly written on an f plane (f constant) as

$$\frac{du}{dt} - fv = M$$
(5a),(5b)
$$\frac{dv}{dt} + fu = 0.$$

Here (u, v) are the velocity components in the direction of the forcing (-M) and 90° to the right, respectively.⁹ The small-amplitude gyroscope dynamics for fast spin are

$$\frac{d\dot{\theta}}{dt} + \omega_{n}\dot{\phi}\sin\theta_{0} = \omega_{p}\omega_{n}\sin\theta_{0}$$

$$\frac{d\dot{\phi}}{dt}\sin\theta_{0} - \omega_{n}\dot{\theta} = 0,$$
(6a),(6b)

where (θ, ϕ) are angles measuring the position of the tip of the spindle on the sphere it circumscribes (θ is colatitude and ϕ is longitude on this sphere; think of Fig. 6 and see Fig. S1). With the spindle horizontal as in Fig. 7 ($\theta_0 = 90^\circ$) the two systems are related under the exchange

$$(u, v, f, M) \Leftrightarrow (\dot{\theta}, \dot{\phi}, -\omega_n, \omega_n \omega_n).$$
 (7)

Swapping the variables this way switches from one set of equations to the other. The difference of sign

between f and ω_n means that inertial oscillation and nutation have opposite senses. That is, inertial oscillation is clockwise in the Northern Hemisphere (Figs. 2, 4), whereas nutation is counterclockwise (Figs. 6, 7).

When the spindle points up ($\theta_0 < 90^\circ$) there are some minor geometric differences in the equations, but the physical analogy remains. In fact, the physical analogy remains under more general conditions too. Both the assumptions of fast spin and smallamplitude oscillations can be relaxed. Then, the gyroscope dynamics coincide with the longwave limit of RSW motion on the sphere. There is a slight difference in a geometrical prefactor in the RSW Coriolis term (see the supplement). This prefactor difference means that the gyroscope exhibits no analogy to the beta effect, which is due to variations in *f* with latitude.

The analogy extends beyond geostrophy and inertial oscillation to related phenomena. Geostrophic adjustment can be illustrated with a gyroscope, for example. Geostrophic adjustment concerns the transient RSW response to an impulse or to an arbitrary initial condition. In general, geostrophic and inertial oscillations are excited, plus some inertia-gravity waves. After several rotation periods the inertial oscillations decay because of weak dissipative processes (which are ignored in the equations above; the inertia-gravity waves propagate into the far field and/or get damped). Geostrophic flow then persists on its own. This process can be seen in Fig. 4 and the inertial oscillation movie. Physically analogous adjustment occurs in the gyroscope. Arbitrary initial conditions excite both nutation and precession. After several rotation periods the nutation decays, however, because of friction (Fig. 7). Releasing the gyroscope from rest is the easiest way to show this behavior.

There are further theoretical analogies too. For example, geostrophy is the minimum energy state in linear undamped RSW flow. Likewise, precession is the minimum energy state for the gyroscope. This means that pure precession is the equivalent to balanced flow in rotating fluids, which is an important theoretical concept. Moreover, gyroscope dynamics and the (longwave limit) RSW equations can be derived from closely related Lagrangian functions using a variational method (supplement). This illuminates the basis of RSW dynamics, as well as

⁹ Note that the longwave limit gives vanishing pressure-gradient force. For this reason, the forcing *M* of the longwave RSW equations is not a real pressure-gradient force but some other uniform body force. This nuance does not undermine the physical analogy between the two systems. The supplement gives a more precise and general statement of the relation.

the Coriolis force, in a nontraditional way. Finally, conserved energy and angular momenta play leading roles in the theory governing the dynamics of each system.

DISCUSSION OF THE ANALOGY AND

DIFFERENCES. The RSW/gyroscope analogy is not perfect. Clear and distinct differences exist between the gyroscope and the longwave limit of the RSW system because of the different geometrical prefactor in the Coriolis term mentioned above. Lack of a beta effect in the gyroscope means that there is no westward drift for nutation as there is for inertial oscillation. Also, the RSW system is more complicated than the gyroscope because gravity (and inertia–gravity) waves also exist. They arise because of spatial variations in the fluid. There are no equivalent waves or spatial variations for the gyroscope.¹⁰

There are other limits to the analogy and some misconceptions to be avoided. First, a useful mental picture of fluid motion is to think about infinitesimal moving fluid parcels. The vorticity of each parcel is analogous to the angular velocity vector of a small solid mass. As the parcels tilt and stretch, the vorticity also tilts and stretches as if the vorticity vector itself was attached to neighboring fluid parcels (Vallis 2006). Stretching of neighboring parcels (and hence the vortex line connecting them) amplifies the vorticity. This mental picture is unrelated to the RSW/gyroscope analogy: vortex lines in a fluid are not like little gyroscopes. The gyroscope analogy applies to RSW flow in which fluid parcels do not tilt or stretch.

The fundamental physics of RSW flow and gyroscopes are not the same. Specifically, the role of rotation in each system is different. In RSW flow (and the real ocean and atmosphere) Coriolis forces arise because we measure distance in a coordinate system that moves with the rotating Earth. For the gyroscope, the coordinate system is fixed to nonrotating space (see Fig. S1). This basic difference means that an inertial oscillation, unlike nutation, need not have any angular momentum measured in a nonrotating frame. For example, an ideal parcel pushed southward from Earth's North Pole will undergo inertial oscillations as viewed from Earth's surface. However, from the perspective of an observer in a fixed frame in space, the parcel moves back and forth in a straight line near the pole. Nutation of a gyroscope does not depend on the choice of a rotating reference frame in the same way. Instead, nutation only exists when the gyroscope spins about its axis and possesses angular momentum. Also the (x,y) coordinates for the RSW Eq. (5) are different to the (θ,ϕ) gyroscope angles. In the *f*-plane RSW system, one is free to rotate the horizontal axes: there is no physical distinction between moving east or moving north. The (θ,ϕ) gyroscope angles cannot be exchanged in this manner. Finally, θ (colatitude) and ϕ (longitude) are restricted to certain ranges because they are angles: the (x,y) coordinates may or may not be restricted, depending on the geometry of the RSW domain.

Geostrophy and inertial oscillation are analogs of gyroscopic precession and nutation in the following sense: there is a correspondence and partial similarity between the phenomena, although they are essentially different. Analogies of this type are common in oceanic and atmospheric science. For instance, the notion of "potential vorticity substance" (Haynes and McIntyre 1990) draws fruitful analogies between potential vorticity (PV) of a fluid and the concentration of a chemical species. Both quantities satisfy a tracer equation, although PV and chemical concentration are dissimilar in other ways. A second example concerns the Lorenz (1963) equations that were developed as a model for atmospheric convection. The Lorenz model applies to many disparate systems. It has been used as a simple model for El Niño-Southern Oscillation (Vallis 1988), a thermohaline loop oscillator (Malkus 1972), and lasers (Haken 1975). Finally, the idea of a "topographic beta effect" draws analogies between changes in the local Coriolis parameter with latitude and changes in fluid depth. Again, there are important fundamental differences between the two ideas but the analogy is valuable and widely exploited. The correspondence between geostrophy and precession, as well as inertial oscillation and nutation, is of the same type as these examples.

CONCLUSIONS. The RSW/gyroscope analogy is fascinating and thought provoking. It is also potentially useful in demonstrating the dynamics of rotating fluids in various educational settings. At the

¹⁰ Some gyroscopes have modified flywheels with variable moments of inertia and hence more complex behavior. The large gyroscope in the bottom panel of Fig. 5 has a hollow flywheel partially full of liquid, for example (a "hydro gyro"). The liquid causes interesting new oscillations associated with surface waves on the liquid/air interface inside the flywheel. These oscillations are not analogous to spatial variations in RSW flow, however.

most basic level, one can simply show that a gyroscope will not fall over when it spins. This strange behavior is similar in oceanic and atmospheric weather systems because the Earth spins. They are strange for similar reasons: in both cases, rotation is a controlling effect. Students at virtually any level can also carefully push on a spinning gyroscope. The spindle slips sideways according to the force and inversely with the spin. The push of the finger is analogous to the pressuregradient force in geostrophy and the opposing push of the gyroscope is analogous to the Coriolis force. This is a profound experience, even for a seasoned instructor, whether in an advanced dynamics course or when illustrating basic principles of motion for precollege-level students. The brass lecture gyroscope in Fig. 5 spinning 100 times per second is particularly memorable.

To this end we have used gyroscope demonstrations in kindergartens and elementary schools and with high-school science teachers, undergraduate humanities students, graduate students in rotating fluid dynamics, and senior faculty in oceanic and atmospheric science. The anecdotal response has been positive, although we have no decisive evidence as to their pedagogical effectiveness. The mathematical analysis supports the demonstration but is often not needed.

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APPENDIX. *Auxiliary Materials.* Five items are included as auxiliary materials. They are as follows:

- Supplement to "Analogies of ocean/atmosphere rotating fluid dynamics with gyroscopes: Teaching opportunities," which contains a technical discussion of gyroscope dynamics and mathematical details of the links to the RSW system.
- 2) geostrophic_movie.mov, a movie showing geostrophic flow in a rotating tank of water, as in Fig. 3. The movie consists of a sequence of still images taken by a camera mounted above the tank and rotating with the tank. There is an image about every 3 s (see the time indicated in the bottom left of the frame). The tank contains

water 10 cm deep, rotating at 20 rpm. The frame size is 18.5 by 12.5 cm; the tank itself is square with side 40 cm. The water is gently stirred (off frame) to produce a turbulent flow that is in geostrophic balance, similar to the ocean eddies shown in Fig. 1. The red and blue colors are food dyes that move with the flow. The black dot is a floating piece of paper from a hole punch.

- 3) inertial_osc_movie.m4v, a movie of inertial oscillation in a rotating tank of water, as in Fig. 4. See the description of the geostrophic movie for more details.
- 4) gyroscope_schematic_movie.m4v, a movie illustrating the fundamental gyroscopic modes: precession (under a torque), nutation (torque-free precession), and both precession and nutation. See also Fig. 6.
- 5) gyroscope_movie.m4v, a slow-motion clip of the MITAC gyroscope undergoing nutation and precession, similar to the top panel of Fig. 7.

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