Non-hydrostatic rotating internal waves

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This worksheet draws heavily from Pedlosky (2003) Ch. 11.

1. Start with the *inviscid, Boussinesq, stratified, linearized* equations for an *unbounded* ocean.

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial P}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial P}{\partial y} \tag{2}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z} - g\frac{\rho}{\rho_0} \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{g}{\rho_0}\frac{\partial \rho}{\partial t} - wN^2 = 0 \tag{5}$$

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}, \qquad P = p/\rho_0 \tag{6}$$

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2. Your goal here is to form an equation in w. Make sure you use the continuity equation to substitute w_z in for $u_x + v_y$ in the first two steps. Unfortunately, the math is a bit tedious.

   a) Take the curl of equations (1) and (2) to form an equation for vorticity, $\zeta = v_x - u_y$. What is the physical meaning of the resulting equation?

   b) Take the divergence of the equations (1) and (2) to form an equation for $u_x + v_y$. You should get

   $$w_{tz} + f\zeta = \nabla_h^2 P$$

   c) Eliminate ζ.

   d) Eliminate ρ between equations (5) and (3)

3. Now you can eliminate P and after some algebra, you should get the wave equation in w.

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 w\right] + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0 \tag{7}$$

Obtain the plane wave dispersion relation by plugging in the wave solution

$$w \sim \exp\left[i(kx + ly + mz - \omega t)\right].$$

Write it in terms of angle ϕ such that $\tan\phi = m^2/k_H^2 = m^2/(k^2 + l^2)$. Can you recover the non-rotating dispersion relation derived earlier in the term?

4. Use $\sin^2\phi = 1 - \cos^2\phi$ and $N^2 \gg f^2$ generally in the ocean to obtain bounds on valid wave solutions for ω. This is the *internal wave band*.

5. Now look at Pedlosky (2003) Figure 11.1. The bounds on ω have important consequences.

   a) What happens if you force the ocean at an ω that is outside these bounds?

   b) In the real ocean f is a function of latitude. Consider a wave that is $\omega = 1.01f$ moving northward so f is increasing as the wave travels. What might happen?

   c) How about a wave moving downwards into water that is less stratified?

6. Now that you have a dispersion relation, can you calculate group and phase velocities? Write the expressions, no need to substitute in for ω

7. After some algebra and re-orienting so that the x axis is along the wave path $l = 0$ and $k_h = k$, you should get (using $K^2 = m^2 + k^2$)

$$(c_g^x, c_g^z) = \frac{(N^2 - f^2)}{\omega K^4} mk(m, -k) \tag{8}$$

   What can you say about the signs of vertical phase speed c_p^z and vertical group velocity c_g^z? Do the same for horizontal phase speeds and group velocities.

8. Mark the vertical directions of phase and energy propagation on Figure 1.

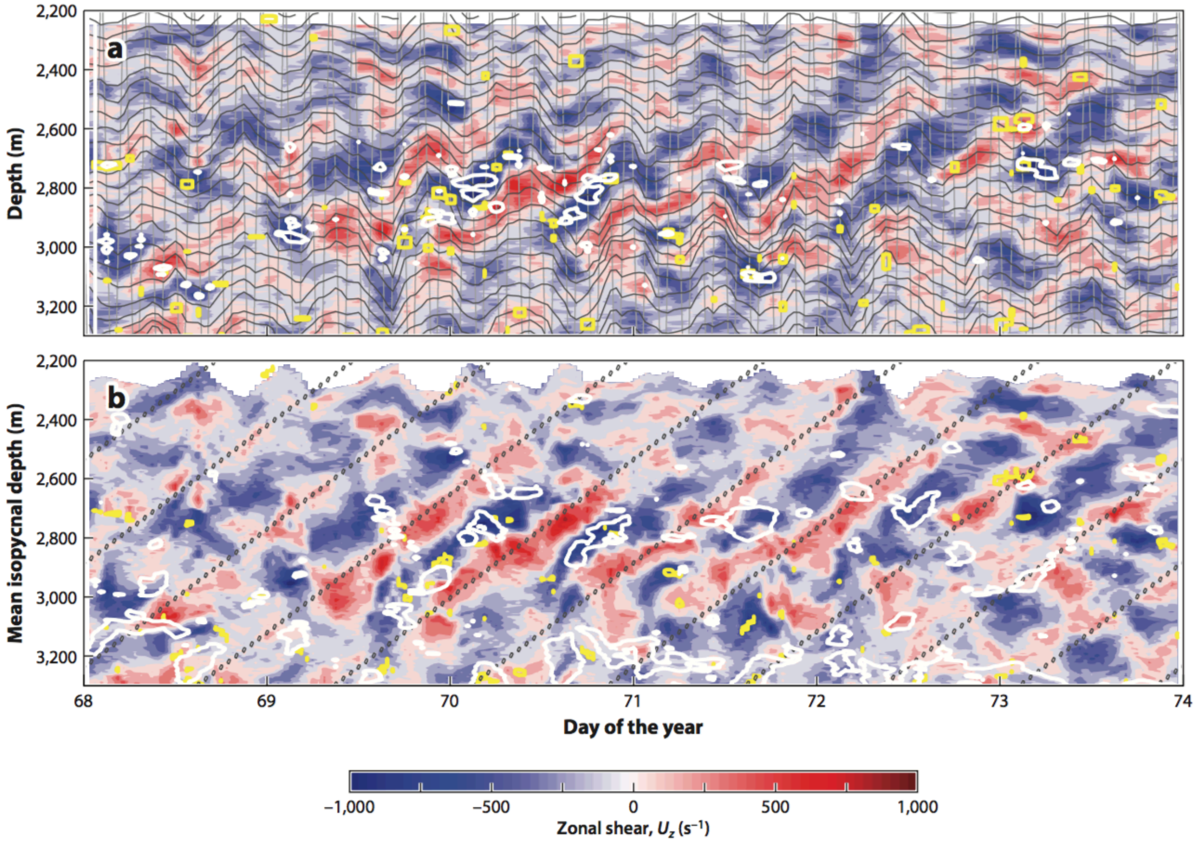

Figure 1: Zonal shear u_z from Alford et al. (2016) - Near Inertial Gravity Waves in the Ocean.

9. Start with this form of the dispersion relation

$$\omega^2 = \frac{f^2 m^2 + N^2 k_H^2}{m^2 + k_H^2}$$

What can you say about typical horizontal and vertical length scales in the ocean? Write that approximation in terms of m and k_H. Use that to simplify this equation.

10. In the *rotated* frame, we can use $w = w_0 \exp\left[i(kx + mz - \omega t)\right]$ and obtain

$$P = -\frac{(N^2 - f^2)}{m\omega} w_0 \exp\left[i(kx + mz - \omega t)\right] \quad (9)$$

$$u = -\frac{m}{k} w_0 \exp\left[i(kx + mz - \omega t)\right] \quad (10)$$

Note that there are no variations in y, $\partial_y \alpha = 0$ for any wave quantity α. Is $v = 0$?

11. What does the sum $u^2 + v^2$ tell you about the variation of the velocity vector with time?

12. So far, we have ignored boundaries. Consider an ocean bounded between a rigid lid and flat bottom. What constraints does this place on the dispersion relation you derived in step 3?

13. Key points:

    a) A lot of characteristics from the non-rotating case carry over to rotating internal waves.

    b) The biggest difference is the introduction of a lower bound on frequency — f. The upper bound N stays the same in both rotating and non-rotating cases. This means that the ocean has an *internal wave band*.